A continuous version of Gale's feasibility theorem(Discrete and Continuous Structures in Optimization)

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数理解析研究所講究録
945 巻 1996 年 83-92

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1. Introduction

There are several approaches to formulate flow problems on continuous networks. In this paper, using a formulation due to Iri (1979) and Strang (1983), we establish a continuous version of Gale's feasibility theorem [1].

The theorem is known as the "Supply - Demand Theorem" in a special case. By means of a cut capacity, this gives a necessary and sufficient condition for an existence of feasible flows.

Let us recall our formulation of continuous network and state a continuous version of the Supply - Demand Theorem. As for a discrete version, one can refer to Ford and Fulderson's book (1962). In this discussion, we assume that all functions and sets are sufficiently smooth. Let Ω be a bounded domain of n-dimensional Euclidean space R^n and $\partial\Omega$ be the boundary. Let A, B be disjoint subsets of $\partial\Omega$ which are regarded as a source and a sink. In our continuous network, every flow is represented by a vector field and every feasible flow σ satisfies the capacity constraint which is written as

$$\sigma(x) \in \Gamma(x) \text{ for all } x \in \Omega,$$

where Γ is a set-valued mapping from Ω to R^n. The flow value of σ is defined by $\sigma \cdot \nu$ on $\partial\Omega$. We call Ω with this capacity constraint a continuous network.

Furthermore, every cut is identified with a subset of Ω in our network. Let S be a cut and ν^S be the unit outer normal to S. Then the cut capacity $C(S)$ is defined by

$$C(S) = \int_{\Omega\cap\partial S} \beta(\nu^S(x), x) ds(x),$$

where

$$\beta(v, x) = \sup_{w\in\Gamma(x)} v \cdot w$$

for $v \in R^n$ and ds is the surface element. If the capacity constraint is isotropic, that is, $\Gamma(x) = \{w \in R^n | \ |w| \leq c(x)\}$ with some nonnegative function $c(x)$, then

$$C(S) = \int_{\Omega\cap\partial S} c(x) ds(x).$$

Let a, b be real-valued functions on A, B respectively and let ν be the unit outer normal to Ω. Then the problem of supply-demand in a simple case is stated as follows:

(SD) Find σ such that
$\sigma(x) \in \Gamma(x)$ for all $x \in \Omega$,
$\operatorname{div} \sigma = 0$ on Ω, $-\sigma \cdot \nu = 0$ on $\partial\Omega - (A \cap B)$,
$-\sigma \cdot \nu \leq a$ on A, $\sigma \cdot \nu \geq b$ on B.

The Supply-Demand theorem assures that (SD) has a solution if and only if

$$\text{(G)} \qquad C(S) \geq \int_{B\cap\partial S} b ds - \int_{A\cap\partial S} a ds \quad \text{for each cut } S.$$

This can be proved by the aid of a continuous version of max-flow min-cut theorem under some assumptions. However, we can not apply the same method to a variant of (SD), which is called a symmetric type by Ford and Fulkerson.

On the other hand, Neumann [5] and Oettli and Yamasaki [8] investigated a problem of feasibility of flows and proved similar results in their own network formulations. Their method is based on a generalized Hahn-Banach Theorem and is applicable even for a symmetric supply-demand problem. In the next section, we give a concrete formulation of our problem in a more general form than (SD), and give a corresponding condition which is equivalent with an existence of solutions for the problem under suitable assumptions. Finally in §3, we consider (SD) as a special case and examine the assumptions.

2. Problem setting and a main theorem

Let Ω be a bounded domain in n-dimensional Euclidean space R^n with Lipschitz boundary $\partial\Omega$. One can consider $n-1$-dimensional surface measure on $\partial\Omega$ which is equal to $n-1$-dimensional Hausdorff measure H_{n-1} on $\partial\Omega$. We note that the unit outer normal ν to Ω is defined and essentially bounded measurable on $\partial\Omega$ with respect to H_{n-1}. Let Γ be a set-valued mapping from Ω to R^n which satisfies the following two conditions:

(H1) $\Gamma(x)$ is a compact convex set containing 0 for all $x \in \Omega$.

(H2) Let $\varepsilon > 0$ and Ω_0 be a compact subset of Ω.
Then there is $\delta > 0$ such that
$\Gamma(x) \subset \Gamma(y) + B(0, \varepsilon)$ if $x, y \in \Omega_0$ and $|x - y| < \delta$.

In what follows, we assume that each feasible flow is represented by an essentially bounded vector field σ on Ω satisfying the following capacity constraints:

$$\sigma(x) \in \Gamma(x) \quad \text{for a.e. } x \in \Omega.$$

Furthermore if $\operatorname{div} \sigma \in L^n(\Omega)$, then $\sigma \cdot \nu$ can be defined as a function in $L^\infty(\partial\Omega)$ in a weak sense by Kohn and Temam [2]. Let $F \in L^n(\Omega)$ and $\lambda, \mu \in L^\infty(\partial\Omega)$ with $\lambda \leq \mu$. Then for the quintuple $(\Omega, \Gamma, F, \mu, \lambda)$, our problem is stated as follows:

(P) Find $\sigma \in L^\infty(\Omega; R^n)$ such that $\sigma(x) \in \Gamma(x)$ for a.e. $x \in \Omega$, $\operatorname{div} \sigma = F$ a.e. on Ω and $\lambda \leq \sigma \cdot \nu \leq \mu$ H_{n-1}-a.e. on $\partial\Omega$

Problem (SD) considered in §1 can be written in this form with $F = 0$.

To specify the class of cuts, we consider the space $BV(\Omega)$ of functions of bounded variation on Ω:

$$BV(\Omega) = \{u \in L^1(\Omega) |\ \nabla u \text{ is a Radon measure of bounded variation on } \Omega\},$$

where $\nabla u = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$ is understood in the sense of distribution. We denote the characteristic function of a subset S of Ω by χ_S and set

$$Q = \{S \subset \Omega |\ \chi_S \in BV(\Omega)\}.$$

Let $S \in Q$. Then the reduced boundary $\partial^* S$ of S is the set of all $x \in \partial S$ where Federer's normal $\nu = \nu(x)$ to S exists. It is known that $\partial^* S$ is a measurable set with respect to both the measure of total variation of $|\nabla \chi_S|$ and H_{n-1}, $|\nabla \chi_S|(R^n - \partial^* S) = 0$ and $|\nabla \chi_S|(E) = H_{n-1}(E)$ for each $|\nabla \chi_S|$-measurable subset E of $\partial^* S$. Furthermore let $\gamma u \in L^1(\partial\Omega)$ be the trace of $u \in BV(\Omega)$. Then [4; Theorem 6.6.2] implies that $\gamma\chi_S = \chi_{\partial^* S \cap \partial\Omega}$ H_{n-1}-a.e. on $\partial\Omega$. Accordingly, replacing ds by H_{n-1} and ∂S by $\partial^* S$, we can define the cut capacity as follows:

$$C(S) = \int_{\Omega \cap \partial^* S} \beta(\nu^S(x), x) dH_{n-1},$$

where $\beta(\cdot, x)$ is the support functional of $\Gamma(x)$ as defined in §1. Let $\nabla u / |\nabla u|$ be the Radon-Nikodym derivative of ∇u with respect to $|\nabla u|$ and set

$$\psi(u) = \int_\Omega \beta(\nabla u / |\nabla u|, x) d|\nabla u|(x)$$

for $u \in BV(\Omega)$. Then $C(S) = \psi(\chi_S)$. Since β is continuous and nonnegative by (H1) and (H2), $C(S)$ is finite. We set

$$\lambda(S) = \int_{\partial\Omega\cap\partial^* S} \lambda dH_{n-1},\ \mu(S) = \int_{\partial\Omega\cap\partial^* S} \mu dH_{n-1},\ F(S) = \int_S F dx.$$

for convenience sake, and consider the condition

(C) $\quad C(S) \geq \lambda(S) - F(S)$ and $C(S) \geq -\mu(\Omega - S) + F(\Omega - S)$ hold for all $S \in Q$.

Now we can state a continuous version of Gale's feasibility theorem.

THEOREM 2.1. *Assume that (H1) and (H2) hold. If (P) has a solution, then condition (C) holds. Conversely if* $\cup_{x\in\Omega}\Gamma(x)$ *is bounded and condition (C) holds, then (P) has a solution.*

To prove this theorem, we need some lemmas. First applying an isoperimetric inequality due to [4] we have

LEMMA 2.2. *There is* $\sigma_0 \in L^\infty(\Omega; R^n)$ *such that* $div\sigma_0 = F$ a.e. *on* Ω.

PROOF: First assume that $\int_\Omega F dx = 0$. We use a max-flow min-cut theorem of Strang's type (1983):

$$\begin{aligned} &\sup\{t \geq 0 \mid \operatorname{div}\sigma = -tF \text{ a.e. on } \Omega,\ \sigma\cdot\nu = 0\ H_{n-1}\text{-a.e. on } \partial\Omega \\ &\qquad\qquad \text{for some } \sigma \in L^\infty(\Omega; R^n) \text{ with } \|\sigma\|_\infty \leq 1\} \\ &= \inf\{H_{n-1}(\Omega\cap\partial^* S)/\int_S F dx \mid \int_S F dx > 0,\ S \subset \Omega, \chi_S \in BV(\Omega)\}. \end{aligned}$$

(The proof is in [6].) To prove the existence of σ_0, it is sufficient to show that the supremum is positive. We can prove that the infimum is positive as follows. According to [4; p.303] there is a positive constant k such that $\min(m_n(S), m_n(\Omega - S)) \leq kH_{n-1}(\Omega\cap\partial^* S)^{n/(n-1)}$, where m_n denotes the Lebesgue measure on R^n. Since

$$\int_S F dx \leq (\int_S 1dx)^{(n-1)/n}\cdot(\int_S |F|^n dx)^{1/n} \leq \|F\|_n(m_n(S))^{(n-1)/n}$$

and

$$\begin{aligned} \int_S F dx = \int_{\Omega - S} -F dx &\leq (\int_{\Omega - S} 1dx)^{(n-1)/n}\cdot(\int_{\Omega - S} |F|^n dx)^{1/n} \\ &\leq \|F\|_n(m_n(\Omega - S))^{(n-1)/n}, \end{aligned}$$

we can conclude that

$$\int_S F dx \leq k_1 H_{n-1}(\Omega \cap \partial^* S)$$

with $k_1 = \|F\|_n k^{(n-1)/n}$ for all $S \in Q$. It follows that the infimum is not less than $1/k_1$.

Finally in case of $\int_\Omega F dx \neq 0$, consider σ_1 such that $\operatorname{div} \sigma_1$ equals constantly $\int_\Omega F dx$, σ_2 such that $\operatorname{div} \sigma_2 = F - \int_\Omega F dx$ and set $\sigma_0 = \sigma_1 + \sigma_2$. Then $\operatorname{div} \sigma_0 = F$. This completes the proof.

From now on we fix σ_0 in Lemma 2.2. For $\sigma \in L^\infty(\Omega; R^n)$ such that $\operatorname{div} \sigma \in L^n(\Omega)$ and $u \in BV(\Omega)$, according to [2] we can define the distribution $(\sigma \nabla u)$ by

$$(\sigma \nabla u)(\varphi) = -\int_\Omega u \nabla \varphi \cdot \sigma dx - \int_\Omega u \varphi \operatorname{div} \sigma dx$$

for $\varphi \in C_0^\infty(\Omega)$. Since $BV(\Omega) \subset L^{n/(n-1)}(\Omega)$, each integral in the definition is finite. Furthermore it is known that $(\sigma \nabla u)$ is regarded as a bounded measure and that

$$(\sigma \nabla u)(\Omega) + \int_\Omega u \operatorname{div} \sigma dx = \int_{\partial\Omega} \gamma u \sigma \cdot \nu dH_{n-1}$$

holds. This is Green's formula due to Kohn and Temam [2; Proposition 1.1]. (See also [6; Theorem 2.3].) Using this formula, we can prove

LEMMA 2.3. *If (P) has a solution, then (C) holds.*

PROOF: Let σ be a solution of (P). Then by Green's formula stated above,

$$\begin{aligned} C(S) &\geq (\sigma \nabla \chi_S)(\Omega) = \int_{\partial\Omega \cap \partial^* S} \sigma \cdot \nu dH_{n-1} - \int_S \operatorname{div} \sigma dx \\ &\geq \lambda(S) - F(S). \end{aligned}$$

Another inequality in (C) can be similarly proved.

To prove the converse, we follow the idea in [5] and [8]. Let us consider the Sobolev space

$$W^{1,1}(\Omega) = \{u \in L^1(\Omega) \mid \nabla u \in L^1(\Omega; R^n)\},$$

which is a linear subspace of $BV(\Omega)$. We set

$$U = L^1(\Omega; R^n) \times L^1(\partial\Omega) \text{ and } V = \{(\nabla u, \gamma u) \mid u \in W^{1,1}(\Omega)\}.$$

Since $\gamma u \in L^1(\partial\Omega)$ for $u \in W^{1,1}(\Omega)$, V is a linear subspace of U. Let $u^+ = \max(u,0)$ and $u^- = -\min(u,0)$. Note that $u^+, u^- \in W^{1,1}(\Omega)$. We define a functional Φ on V by

$$\begin{aligned}\Phi(\nabla u, \gamma u) = &\int_\Omega \sigma_0 \cdot \nabla u dx - \int_{\partial\Omega} \sigma_0 \cdot \nu\gamma u dH_{n-1} \\ &+ \int_{\partial\Omega} \lambda\gamma u^+ dH_{n-1} - \int_{\partial\Omega} \mu\gamma u^- dH_{n-1}\end{aligned}$$

and set

$$K = \{\sigma \in L^\infty(\Omega; R^n) | \ \sigma(x) \in \Gamma(x) \text{ for a.e. } x \in \Omega\}.$$

For $v \in L^1(\Omega; R^n)$, we define a functional ρ on U by

$$\rho(v,\alpha) = \int_\Omega \beta(v(x), x) dx = \sup_{\phi \in K} \int_\Omega v \cdot \phi dx$$

for $(v,\alpha) \in U$. The last equality follows from a measurable selection theorem. (Cf. Castaing and Valadier (1977).) Since $\rho(v,\alpha)$ is independent of α, it is sometimes denoted by $\rho(v)$. We note that $\psi(u) = \rho(\nabla u)$ for all $u \in W^{1,1}(\Omega)$. The inequality $\lambda \leq \mu$ implies the next lemma.

LEMMA 2.4. *Φ is superlinear on V, that is , concave and positively homogeneous, and ρ is sublinear on U, that is, $-\rho$ is superlinear. Furthermore ρ is continuous at the origin of U if $\cup_{x\in\Omega}\Gamma(x)$ is bounded.*

Condition (C) can be replaced by an inequality with Φ and ρ.

LEMMA 2.5. *If (C) holds, then $\Phi \leq \rho$ on V.*

PROOF: We use equalities of coarea formula type which are stated in [6]: Let $u \in W^{1,1}(\Omega)$. Set $N_t = \{x \in \Omega | \ u(x) \geq t\}$ and $M_t = \Omega - N_t$ for any real number t. Then $N_t, M_t \in Q$ for a.e. t and

$$\psi(u) = \int_{-\infty}^{\infty} \psi(\chi_{N_t}) dt.$$

Furthermore by [6; Lemma 4.6]

$$\int_\Omega F u dx = \int_0^\infty \left(\int_\Omega F\chi_{N_t} dx - \int_\Omega F\chi_{M_{-t}} dx\right) dt,$$

$$\int_{\partial\Omega} \lambda\gamma u^+ dH_{n-1} = \int_0^\infty \int_{\partial\Omega} \lambda\gamma\chi_{N_t} dH_{n-1} dt,$$

$$\int_{\partial\Omega} \mu\gamma u^- dH_{n-1} = \int_0^\infty \int_{\partial\Omega} \mu\gamma\chi_{M_{-t}} dH_{n-1} dt.$$

It follows from these equalities and (C) that

$$\begin{aligned}
\rho(\nabla u) = \psi(u) &= \int_{-\infty}^{\infty} \psi(\chi_{N_t})dt = \int_0^{\infty} \psi(\chi_{N_t})dt + \int_0^{\infty} \psi(\chi_{\Omega-M_{-t}})dt \\
&= \int_0^{\infty} C(N_t)dt + \int_0^{\infty} C(\Omega - M_{-t})dt \\
&= \int_0^{\infty} (\lambda(N_t) - F(N_t))dt + \int_0^{\infty} (-\mu(M_{-t}) + F(M_{-t}))dt \\
&\geq \int_0^{\infty} (\int_{\partial\Omega} \lambda\gamma\chi_{N_t} dH_{n-1} - \int_\Omega F\chi_{N_t} dx)dt \\
&\quad + \int_0^{\infty} (-\int_{\partial\Omega} \mu\gamma\chi_{M_{-t}} dH_{n-1} + \int_\Omega F\chi_{M_{-t}} dx)dt \\
&= \int_{\partial\Omega} \lambda\gamma u^+ dH_{n-1} - \int_{\partial\Omega} \mu\gamma u^- dH_{n-1} - \int_\Omega u \operatorname{div} \sigma_0 dx \\
&= \int_{\partial\Omega} \lambda\gamma u^+ dH_{n-1} - \int_{\partial\Omega} \mu\gamma u^- dH_{n-1} \\
&\quad - \int_{\partial\Omega} \sigma_0 \cdot \nu\gamma u H_{n-1} + \int_\Omega \sigma_0 \cdot \nabla u dx \\
&\geq \Phi(\nabla u, \gamma u).
\end{aligned}$$

Here we have used Green's formula in the last equality. This completes the proof.

By Lemma 2.5 and a version of Hahn-Banach theorem ([3; Corollary 2.2 in p.114]), there is a linear functional ξ on U satisfying $\Phi \leq \xi$ on V and $\xi \leq \rho$ on U. The next lemma is directly proved.

LEMMA 2.6. *If $\cup_{x\in\Omega}\Gamma(x)$ is bounded, then ξ is continuous on U with respect to the canonical norm topology.*

By Lemma 2.6, there is $\sigma \in L^\infty(\Omega; R^n)$ and $\eta \in L^\infty(\partial\Omega)$ such that

$$\xi(v, \alpha) = \int_\Omega \sigma \cdot v dx + \int_{\partial\Omega} \eta\alpha dH_{n-1}$$

for all $(v, \alpha) \in U$. However, from the inequality $\xi(v, \alpha) \leq \rho(v)$ for all $\alpha \in L^\infty(\partial\Omega)$, η must be 0.

LEMMA 2.7. *Assume that* $\cup_{x\in\Omega}\Gamma(x)$ *is bounded. Then the vector field* σ *obtained above is a solution to (P).*

PROOF: We set $\Omega_0 = \{x \in \Omega |\ 0 \notin \Gamma(x) - \sigma(x)\}$. Then Ω_0 is a measurable set. Assume that the measure of Ω_0 is positive. Since $\hat{K} = \{\phi \in L^\infty(\Omega; R^n)|\ \phi(x) \in \Gamma(x) - \sigma(x)\}$ is a weakly* closed convex set and does not contain 0, there is $\varphi \in L^1(\Omega; R^n)$ such that $\sup_{\phi\in\hat{K}} \int_\Omega \varphi \cdot \phi dx < 0$. Therefore

$$\rho(\varphi) = \sup_{\phi\in\hat{K}} \int_\Omega \varphi \cdot (\phi + \sigma) dx < \int_\Omega \varphi \cdot \sigma dx = \xi(\varphi, 0).$$

This is a contradiction since $\xi \leq \rho$ on U. Thus $\sigma(x) \in \Gamma(x)$ for almost all $x \in \Omega$.

Next we prove $\operatorname{div} \sigma = F$. If $u \in C_0^\infty(\Omega)$, then $\gamma u = 0$ so that

$$\Phi(\nabla u, \gamma u) = \int_\Omega \sigma_0 \cdot \nabla u dx \leq \xi(\nabla u, 0) = \int_\Omega \sigma \cdot \nabla u dx.$$

It follows that

$$\int_\Omega \sigma_0 \cdot \nabla u dx = \int_\Omega \sigma \cdot \nabla u dx$$

for all $u \in C_0^\infty(\Omega)$. This implies that $\operatorname{div} \sigma = \operatorname{div} \sigma_0 = F$ in a distribution sense.

Finally we prove that $\lambda \leq \sigma \cdot \nu \leq \mu$ H_{n-1}-a.e. on $\partial\Omega$. Since $\operatorname{div} \sigma = F \in L^n(\Omega)$, $\sigma \cdot \nu$ is defined as a function in $L^\infty(\partial\Omega)$ and the inequality $\Phi(\nabla u, \gamma u) \leq \int_\Omega \sigma \cdot \nabla u dx$ implies that

$$\int_{\partial\Omega} \lambda\gamma u^+ - \mu\gamma u^- dH_{n-1} \leq \int_{\partial\Omega} \gamma u \sigma \cdot \nu dH_{n-1}.$$

For any $\alpha \in L^1(\partial\Omega)$, there is $u \in W^{1,1}(\Omega)$ such that $\alpha = \gamma u$ by Gagliardo (1957). Thus for any nonnegative function $\alpha \in L^1(\partial\Omega)$, we have

$$\int_{\partial\Omega} \lambda\alpha dx \leq \int_{\partial\Omega} \sigma \cdot \nu\alpha dH_{n-1},$$

$$-\int_{\partial\Omega} \mu\alpha dx \leq -\int_{\partial\Omega} \sigma \cdot \nu\alpha dH_{n-1}.$$

Accordingly, $\lambda \leq \sigma \cdot \nu \leq \mu$ H_{n-1}-a.e. on $\partial\Omega$. This completes the proof.

PROOF OF THEOREM 2.1: The first statement follows from Lemma 2.3 and the second statement follows from Lemma 2.7.

3. Supply - Demand theorem

Let A, B be disjoint Borel subsets of $\partial\Omega$ and a, b be Borel measurable functions on A, B respectively. Then (SD) in §1 should be written in the following concrete form:

$$\text{(SD)} \qquad \begin{aligned} &\text{Find } \sigma \in L(\Omega; R^n) \\ &\text{such that } \sigma(x) \in \Gamma(x) \quad \text{for a.e. } x \in \Omega, \\ &\operatorname{div}\sigma = 0 \text{ a.e. on } \Omega, \\ &\sigma \cdot \nu = 0 \ H_{n-1}\text{-a.e. on } \partial\Omega - (A \cap B), \\ &-\sigma \cdot \nu \leq a \quad H_{n-1}\text{-a.e. on } A, \\ &\sigma \cdot \nu \geq b \quad H_{n-1}\text{-a.e. on } B. \end{aligned}$$

By setting $\lambda = -a$ on A, $\lambda = b$ on B , $\lambda = 0$ elsewhere on $\partial\Omega$ and $\mu = \max(\lambda, 0)$, Theorem 2.1 implies

THEOREM 3.1. *Assume that (H1), (H2) hold and that $\cup_{x\in\Omega}\Gamma(x)$ is bounded. Then (SD) has a solution if and only if*

$$\text{(G)} \qquad C(S) \geq \int_{B\cap\partial^* S} b\,dH_{n-1} - \int_{A\cap\partial^* S} a\,dH_{n-1} \text{ for all } S \in Q.$$

Finally we refer to a relation between (SD) and a max-flow problem of Strang's type (MFS) which has been used in the proof of Lemma 2.2 with the boundary condition $\sigma \cdot \nu = 0$. Now let f be an arbitrary function in $L^\infty(\partial\Omega)$ which satisfies the conservation law $\int_{\partial\Omega} f\,dH_{n-1} = 0$. Then for (Ω, Γ, f) , (MFS) with $F = 0$ is stated as follows:

$$\text{(MFS)} \qquad \begin{aligned} &\text{Maximize } \lambda \\ &\text{subject to } (\lambda, \sigma) \in R \times L^\infty(\Omega; R^n), \\ &\sigma(x) \in \Gamma(x) \text{ a.e. } x \in \Omega, \\ &\operatorname{div}\sigma = 0 \text{ a.e. on } \Omega, \ \sigma \cdot \nu = \lambda f \text{ a.e. on } \partial\Omega, \end{aligned}$$

and the corresponding min-cut problem (MCS) is

$$\text{(MCS)} \qquad \begin{aligned} &\text{Minimize } C(S)/L(S) \\ &\text{subject to } S \subset \Omega, \chi_S \in BV(\Omega), L(S) > 0, \end{aligned}$$

where $L(S) = \int_{\partial\Omega\cap\partial^* S} f\,dH_{n-1}$. Then we have

Proposition 3.2. *Assume that (H1) and (H2) hold.*

(1) Assume that (G) implies the existence of solutions to (SD) for any disjoint Borel subsets A, B *of* $\partial\Omega$ *and* $a \in L^\infty(A), b \in L^\infty(B)$*. Then* $MFS = MCS$ *and (MFS) has an optimal solution for any* $f \in L^\infty(\partial\Omega)$ *satisfying the conservation law.*

(2) Conversely if $MFS = MCS$ *and (MFS) has an optimal solutin for any* $f \in L^\infty(\partial\Omega)$ *satisfying the conservation law, then (G) implies the existence of solutions to (SD) for any disjoint Borel subsets* A, B *of* $\partial\Omega$ *and* $a \in L^\infty(A)$*,* $b \in L^\infty(B)$ *such that* $\int_A a dH_{n-1} = \int_B b dH_{n-1}$.

It is known that there is an example with $MFS < MCS$ if Γ is unbounded. (See [7].) Thus Proposition 3.2 (1) shows that there is an example of (SD) such that $\cup_{x\in\Omega}\Gamma(x)$ is bounded, condition (G) is satisfied and (SD) has no solution.

Acknowledgement

The author is grateful to Professor Yamasaki for his valuable advise, which is essential in proving Theorem 2.1.

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